Casualty Actuarial Society
Dynamic Financial Analysis Seminar

LIABILITY DYNAMICS

Stephen Mildenhall
CNA Re
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Objectives

- Illustrate some liability modeling concepts
  - General comments
  - Efficient use of simulation
  - How to model correlation
  - Adding distributions using Fourier transforms
  - Case Study to show practical applications
- Emphasis on practice rather than theory
  - Actuaries are the experts on liability dynamics
  - Knowledge is not embodied in general theories
  - Techniques you can try for yourselves

General Comments

- Importance of liability dynamics in DFA models
  - Underwriting liabilities central to an insurance company; DFA models should reflect this
  - DFA models should ensure balance between asset and liability modeling sophistication
  - Asset models can be very sophisticated
    - Don't want to change investment strategy based on half-baked liability model
  - Need clear idea of what you are trying to accomplish with DFA before building model

General Comments

- Losses or Loss Ratios?
  - Must model two of premium, losses, and loss ratio
  - Ratios harder to model than components
    - Ratio of independent normals is Cauchy
  - Model premium and losses separately and compute loss ratio
    - Allows modeler to focus on separate drivers
    - Liability: inflation, econometric measures, gas prices
    - Premiums: pricing cycle, industry results, cat experience
    - Explicitly builds in structural correlation between lines driven by pricing cycles

General Comments

- Aggregate Loss Distributions
  - Determined by frequency and severity components
  - Tail of aggregate determined by thicker of the tails of frequency and severity components
  - Frequency distribution is key for coverages with policy limits (most liability coverages)
  - Cat losses can be regarded as driven by either component
    - Model on a per occurrence basis: severity component very thick tailed, frequency thin tailed
    - Model on a per risk basis: severity component thin tailed, frequency thick tailed
  - Focus on the important distribution!

General Comments

- Loss development: resolution of uncertainty
  - Similar to modeling term structure of interest rates
  - Emergence and development of losses
  - Correlation between development between lines and within a line between calendar years
  - Very complex problem
  - Opportunity to use financial market's techniques
- Serial correlation
  - Within a line (1995 results to 1996, 1996 to 1997 etc.)
  - Between lines
  - Calendar versus accident year viewpoints

Efficient Use of Simulation

- Monte Carlo simulation essential tool for integrating functions over complex regions in many dimensions
- Typically not useful for problems only involving one variable
  - More efficient routines available for computing one-dimensional integrals
- Not an efficient way to add up, or convolve, independent distributions
  - See below for alternative approach

Efficient Use of Simulation

- Example
  - Compute expected claim severity excess of $100,000 from lognormal severity distribution with mean $30,000 and CV = 3.0
  - Comparison of six methods

Method	Estimate	% Error
100 random points	N/A	Too high
100 random points xs $100,000	N/A	> 25% common
99 percentiles xs $100,000	$7,713	-9.8%
Newton-Coates using 99 points xs $100,000	$8,199	-4.1%
Gauss-Legendre, 10 points xs $100,000	$8,173	-4.4%
Gauss-Legendre, 20 points xs $100,000	$8,403	-1.8%
Exact solution from analytic formula	$8,553	

Efficient Use of Simulation

- Comparison of Methods
  - Not selecting xs $100,000 throws away 94% of points
  - Newton-Coates is special weighting of percentiles
  - Gauss-Legendre is clever weighting of cleverly selected points
    - See 3C text for more details on Newton-Coates and Gauss-Legendre
  - When using numerical methods check hypotheses hold
    - For layer $900,000 excess of $100,000 Newton-Coates outperforms Gauss-Legendre because integrand is not differentiable near top limit
- Summary
  - Consider numerical integration techniques before simulation, especially for one dimensional problems
  - Concentrate simulated points in area of interest

Correlation

- S. Wang, *Aggregation of Correlated Risk Portfolios: Models and Algorithms*
  - http://www.casact.org/cotor/wang.htm
- Measures of correlation
  - Pearson's correlation coefficient
    - Usual notion of correlation coefficient, computed as covariance divided by product of standard deviations
    - Most appropriate for normally distributed data
  - Spearman's rank correlation coefficient
    - Correlation between ranks (order of data)
    - More robust than Pearson's correlation coefficient
  - Kendall's tau

Correlation

- Problems with modeling correlation
  - Determining correlation
    - Typically data intensive, but companies only have a few data points available
    - No need to model guessed correlation with high precision
  - Partial correlation
    - Small cats uncorrelated but large cats correlated
    - Rank correlation and Kendall’s tau less sensitive to partial correlation

Correlation

- Problems with modeling correlation
  - Hard to simulate from multivariate distributions
    - E.g. Loss and ALAE
    - No analog of using $F^{-1}(u)$ where u is a uniform variable
    - Can simulate from multivariate normal distribution
  - DFA applications require samples from multivariate distribution
    - Sample essential for loss discounting, applying reinsurance structures with sub-limits, and other applications
    - Samples needed for Monte Carlo simulation

Correlation

- What is positive correlation?
  - The tendency for above average observations to be associated with other above average observations
  - Can simulate this effect using "shuffles" of marginals
  - Vitale's Theorem
    - Any multivariate distribution with continuous marginals can be approximated arbitrarily closely by a shuffle
  - Iman and Conover describe an easy-to-implement method for computing the correct shuffle
    - *A Distribution-Free Approach to Inducing Rank Correlation Among Input Variables*, Communications in Statistical Simulation & Computation (1982) 11(3), p. 311-334

Correlation

- Advantages of Iman-Conover method
  - Easy to code
  - Quick to apply
  - Reproduces input marginal distributions
  - Easy to apply different correlation structures to the same input marginal distributions for sensitivity testing

Correlation

- How Iman-Conover works
  - Inputs: marginal distributions and correlation matrix
  - Use multivariate normal distribution to get a sample of the required size with the correct correlation
    - *Introduction to Stochastic Simulation*, 4B syllabus
    - Use Choleski decomposition of correlation matrix
  - Reorder (shuffle) input marginals to have the same ranks as the normal sample
    - Implies sample has same rank correlation as the normal sample
    - Since rank correlation and Pearson correlation are typically close, resulting sample has the desired structure
  - Similar to normal copula method

Adding Loss Distributions

- Using Fast Fourier Transform to add independent loss distributions
  - Method
    - (1) Discretize each distribution
    - (2) Take FFT of each discrete distribution
    - (3) Form componetwise product of FFTs
    - (4) Take inverse FFT to get discretization of aggregate
  - FFT available in SAS, Excel, MATLAB, and others
  - Example on next slide adds independent N(70,100) and N(100,225), and compares results to N(170,325)
    - 512 equally sized buckets starting at 0 (up to 0.5), 0.5 to 1.5,...
    - Maximum percentage error in density function is 0.3%
    - Uses Excel

Adding Loss Distributions

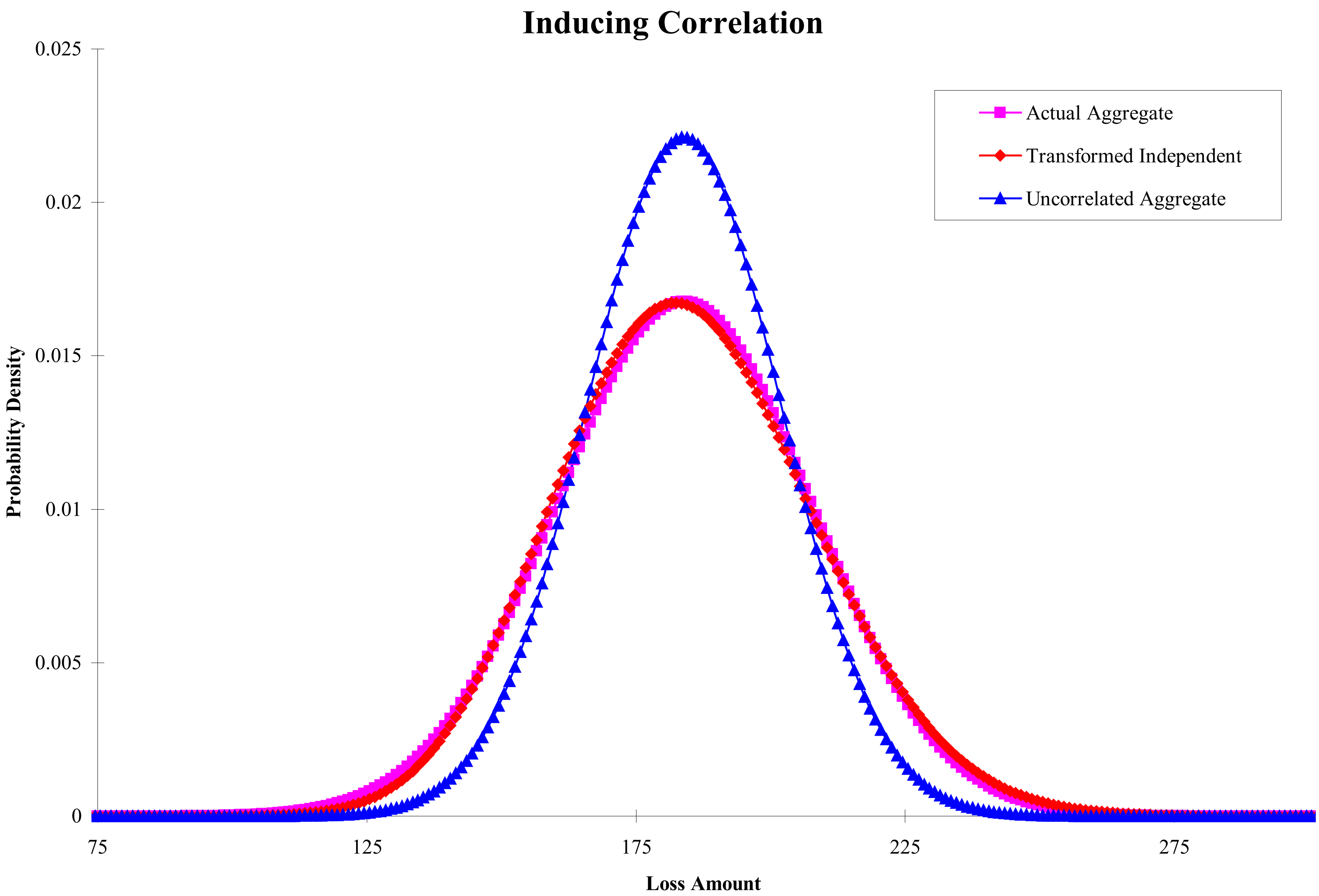
512 rows

Bucket	N(70;100)	N(100;225)	FFT N(70;100)	FFT N(100;225)	Product of FFTs	Inverse FFT	N(170;325)	% Error
<0.5	1.8380E-12	1.6497E-11	1.00	1.00	1.00	0.0000E+00	0.0000E+00	0.00%
0.5-1.5	1.8767E-12	9.3621E-12	0.648-0.752i	0.331-0.926i	-0.481-0.849i	0.0000E+00	0.0000E+00	0.00%
1.5-2.5	3.7196E-12	1.4499E-11	-0.142-0.960i	-0.722-0.593i	-0.466+0.778i	0.0000E+00	0.0000E+00	0.00%
61.5-62.5	2.8965E-02	1.0756E-03	0.000+0.000i	0.000-0.000i	0.000-0.000i	3.5994E-10	3.5895E-10	-0.28%
62.5-63.5	3.1219E-02	1.2706E-03	0.000-0.000i	0.000-0.000i	0.000-0.000i	5.0095E-10	4.9956E-10	-0.28%
63.5-64.5	3.3314E-02	1.4943E-03	0.000-0.000i	0.000-0.000i	0.000+0.000i	6.9506E-10	6.9313E-10	-0.28%
64.5-65.5	3.5196E-02	1.7496E-03	0.000+0.000i	0.000+0.000i	0.000+0.000i	9.6141E-10	9.5874E-10	-0.28%
65.5-66.5	3.6814E-02	2.0394E-03	0.000-0.000i	0.000+0.000i	0.000+0.000i	1.3257E-09	1.3221E-09	-0.28%
66.5-67.5	3.8124E-02	2.3666E-03	0.000-0.000i	0.000-0.000i	0.000-0.000i	1.8225E-09	1.8175E-09	-0.28%
67.5-68.5	3.9089E-02	2.7343E-03	0.000-0.000i	0.000-0.000i	0.000+0.000i	2.4977E-09	2.4909E-09	-0.27%
68.5-69.5	3.9679E-02	3.1450E-03	0.000-0.000i	0.000-0.000i	0.000+0.000i	3.4126E-09	3.4034E-09	-0.27%
69.5-70.5	3.9878E-02	3.6014E-03	0.000+0.000i	0.000+0.000i	0.000-0.000i	4.6482E-09	4.6358E-09	-0.27%
70.5-71.5	3.9679E-02	4.1057E-03	0.000+0.000i	0.000+0.000i	0.000+0.000i	6.3118E-09	6.2952E-09	-0.26%
71.5-72.5	3.9089E-02	4.6600E-03	0.000+0.000i	0.000-0.000i	0.000+0.000i	8.5445E-09	8.5222E-09	-0.26%
72.5-73.5	3.8124E-02	5.2655E-03	0.000+0.000i	0.000-0.000i	0.000-0.000i	1.1531E-08	1.1502E-08	-0.26%
73.5-74.5	3.6814E-02	5.9234E-03	0.000-0.000i	0.000-0.000i	0.000+0.000i	1.5515E-08	1.5475E-08	-0.26%
74.5-75.5	3.5196E-02	6.6340E-03	0.000-0.000i	0.000+0.000i	0.000+0.000i	2.0810E-08	2.0758E-08	-0.25%
127.5-128.5	2.0098E-09	4.6600E-03	0.000-0.000i	0.000-0.000i	0.000-0.000i	1.4684E-03	1.4676E-03	-0.06%
128.5-129.5	1.1204E-09	4.1057E-03	0.000+0.000i	0.000-0.000i	0.000+0.000i	1.6683E-03	1.6674E-03	-0.05%
129.5-130.5	6.1844E-10	3.6014E-03	0.000+0.000i	0.000-0.000i	0.000-0.000i	1.8896E-03	1.8887E-03	-0.05%
130.5-131.5	3.3796E-10	3.1450E-03	0.000+0.000i	0.000+0.000i	0.000-0.000i	2.1337E-03	2.1327E-03	-0.05%
131.5-132.5	1.8286E-10	2.7343E-03	0.000-0.000i	0.000+0.000i	0.000+0.000i	2.4019E-03	2.4008E-03	-0.04%
132.5-133.5	9.7952E-11	2.3666E-03	0.000-0.000i	0.000+0.000i	0.000-0.000i	2.6955E-03	2.6944E-03	-0.04%
133.5-134.5	5.1949E-11	2.0394E-03	0.000-0.000i	0.000-0.000i	0.000-0.000i	3.0157E-03	3.0145E-03	-0.04%
134.5-135.5	2.7278E-11	1.7496E-03	0.000-0.000i	0.000-0.000i	0.000+0.000i	3.3636E-03	3.3624E-03	-0.04%
135.5-136.5	1.4181E-11	1.4943E-03	0.000+0.000i	0.000+0.000i	0.000-0.000i	3.7400E-03	3.7388E-03	-0.03%
136.5-137.5	7.2992E-12	1.2706E-03	0.000+0.000i	0.000+0.000i	0.000-0.000i	4.1459E-03	4.1447E-03	-0.03%
137.5-138.5	3.7196E-12	1.0756E-03	0.000+0.000i	0.000+0.000i	0.000+0.000i	4.5817E-03	4.5805E-03	-0.03%
509.5-510.5	0.0000E+00	0.0000E+00	-0.142+0.960i	-0.722+0.593i	-0.466-0.778i	0.0000E+00	0.0000E+00	0.00%
510.5+	0.0000E+00	0.0000E+00	0.648+0.752i	0.331+0.926i	-0.481+0.849i	0.0000E+00	0.0000E+00	0.00%

Adding Loss Distributions

- Using fudge factor to approximate correlation in aggregates
  - Correlation increases variance of sum
    - Can compute variance given marginals and covariance matrix
  - Increase variance of independent aggregate to desired quantity using Wang's proportional hazard transform, by adding noise, or some other method
  - Shift resulting distribution to keep mean unchanged
- Example, continued
  - If correlation is 0.8, aggregate is N(170,565)
  - Approximation, Wang's rho = 2.3278, shown below

Adding Loss Distributions



DFA Liability Case Study

- Problem
  - Compute capital needed for various levels of one year exp-ected policyholder deficit (EPD) and probability of ruin
- Assumptions
  - Monoline auto liability (BI and PD) company
  - All losses at ultimate after four years
  - Loss trend 5% with matching rate increases
  - Ultimates booked at best estimates
  - Anything else required to keep things simple
    - Expenses paid during year; premiums paid in full during year; no uncollected premium; assets all in cash; ...

DFA Liability Case Study

- Historical results and AY 1998 plan at 12/97

Accident Year	Earned Premium	Paid Loss at 12/97	Estimated Ult Loss at 12/97	Estimated Ult LR at 12/97
1995	11,301,129	7,013,848	7,792,385	69.0%
1996	11,729,399	5,995,292	8,296,654	70.7%
1997	12,363,136	3,353,340	9,087,142	73.5%
1998	12,967,731	0	9,097,549	70.2%

Accident Year	Paid During 1998	Paid Loss at 12/98	Est Ult Loss at 12/98	Alpha	Beta	Shift
1995	778,537	7,792,385	7,792,385	2.8736	270,927.45	7,013,848
1996	1,472,444	7,467,736	8,296,654	4.6477	351,977.32	6,660,769
1997	3,213,170	6,566,510	9,087,142	7.3058	608,637.40	4,640,558
1998	3,357,181	3,357,181	9,097,549	***		

DFA Liability Case Study

- EPD calculation requires distribution of calendar year 1998 incurred loss
- For AY95-97 derive from amounts paid during 98
  - Assume LDFs do not change from current estimate

$$\Delta Ult = LDF_{98}(Paid_{\text{in 1998}} - (link_{98} - 1)Paid_{\text{prior to 1998}})$$

Expected value

Random component

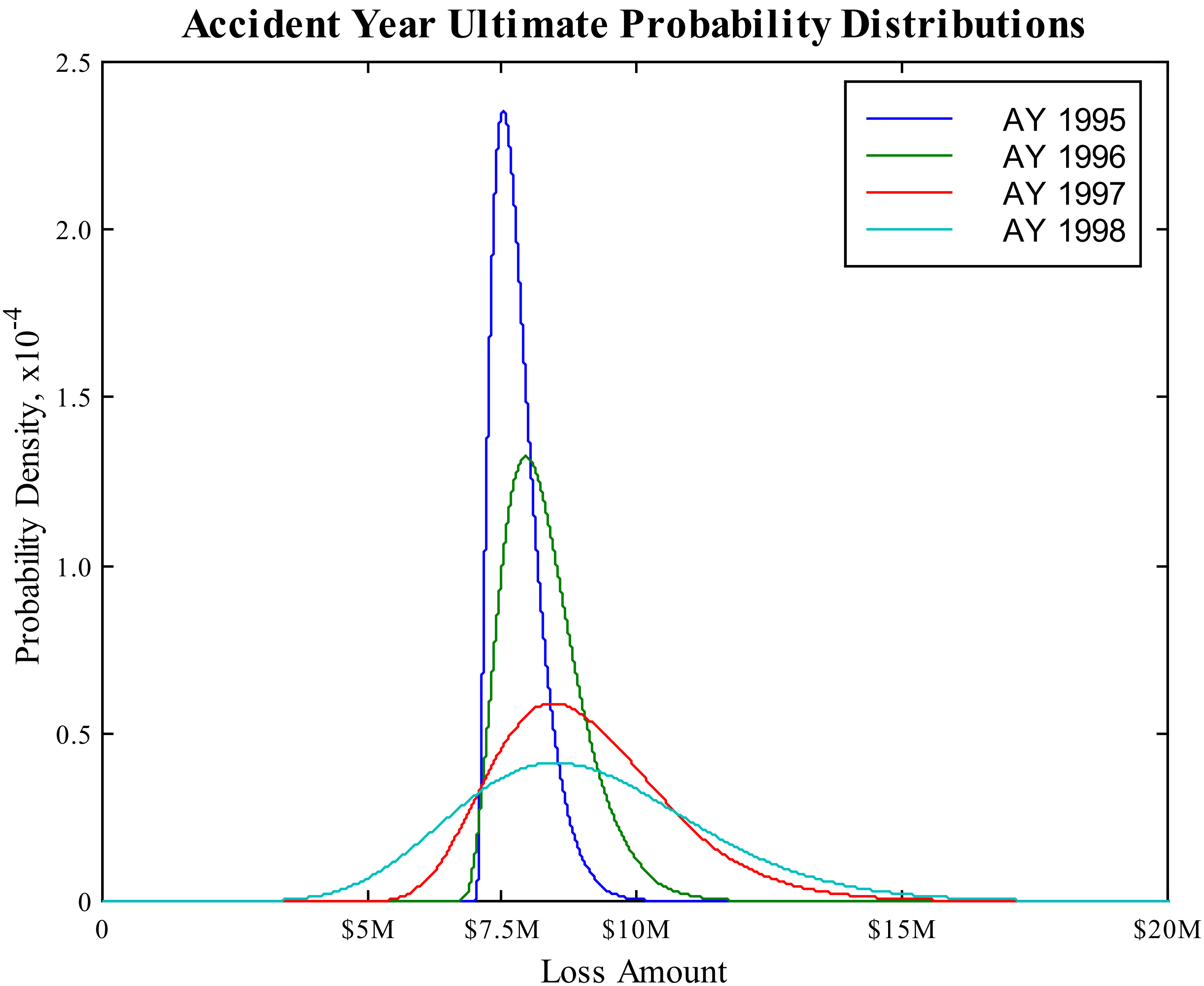
- For AY98 model ultimate using an aggregate loss distribution

DFA Liability Case Study

- Liability model for AY 1997 and prior
  - Used annual statement extract from Private Passenger Auto Liability to generate sample of 344 four-year paid loss triangles
  - Fitted gamma distribution to one-year incremental paid losses
    - New ultimate has shifted gamma distribution, parameters given on page 11
    - Used generalized linear model theory to determine maximum likelihood parameters
  - CV of *reserves* increased with age
  - CV estimates used here exactly as produced by model

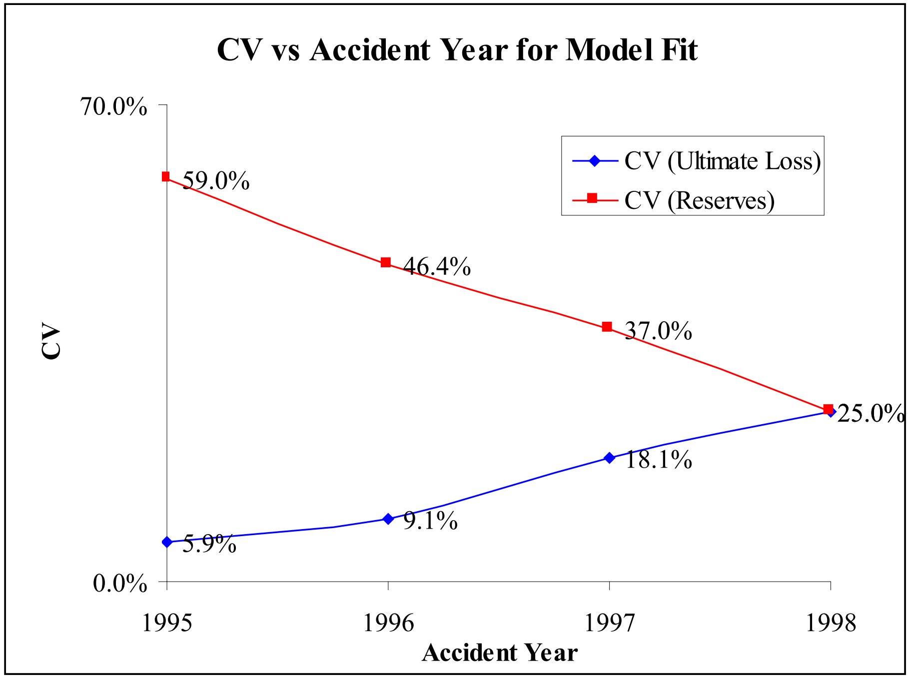
DFA Liability Case Study

- Aggregate liability model for AY 1998
  - Property Damage Severity: lognormal
  - Bodily Injury Severity: ISO Five Parameter Pareto
  - Total Severity: 30% of PD claims lead to BI claims
    - Used FFT to generate total severity
    - Mean severity $2,806 (CV = 1.6, skewness = 2.1)
  - Negative binomial claim count
    - Mean 3,242 (CV=0.25)
  - Computed aggregate using FFT
    - Mean = $9.098M (CV = 0.25, skewness = 0.50)
- Next slide shows resulting marginal distributions


Accident Year Ultimate Probability Distributions
AY 1995
AY 1996
AY 1997
AY 1998
Probability Density, x10^-4
2.5
2.0
1.5
1.0
0.5
0
0
$5M
$7.5M
$10M
$15M
$20M
Loss Amount

DFA Liability Case Study

- Comments
  - Model agrees with *a priori* expectations
  - Single company may not want to base reserve development pattern on other companies
  - Graph opposite shows CV to total loss and reserves
  - See forthcoming Taylor paper for other approaches



CV(Ultimate Loss) = SD(Reserves)/E(Ultimate Loss)

CV(Reserves) = SD(Reserves)/E(Reserves)

E(Ultimate Loss) > E(Reserves)

DFA Liability Case Study

- Correlation
  - Annual Statement data suggested there was a calendar year correlation in the incremental paid amounts
    - Higher than expected paid for one AY in a CY increases likelihood of higher than expected amount for other AYs
    - Some data problems
  - Model with and without correlation to assess impact

DFA Liability Case Study

- EPD calculation
  - 10,000 "0.01%ile" points from each marginal distribution shuffled using Iman-Conover
    - With no correlation could also use FFT to convolve marginal distributions directly
    - Sensitivity testing indicates 10,000 points is just about enough
  - EPD ratios computed to total ultimate losses
  - Exhibits also show premium to surplus (P:S) and liability to surplus ratio (L:S) for added perspective
  - Coded in MATLAB
    - Computation took 90 seconds on Pentium 266 P/C

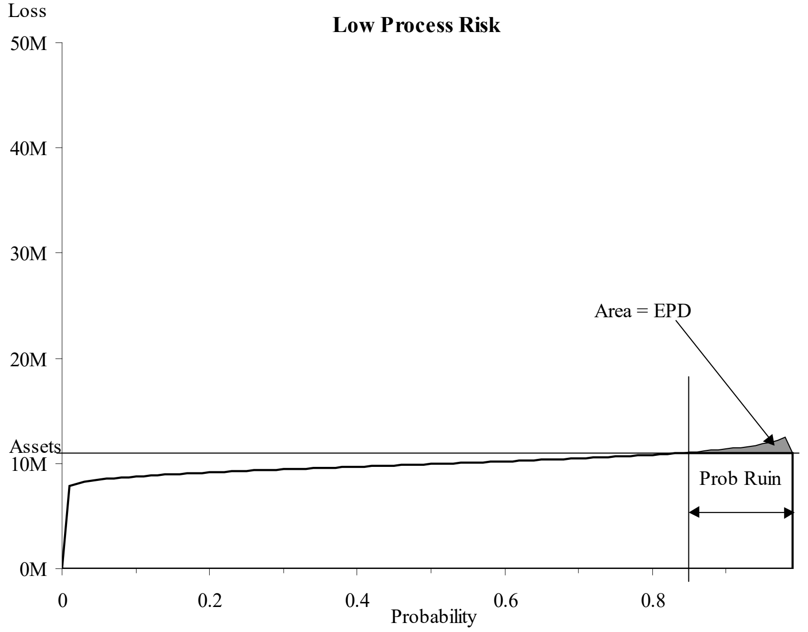
DFA Liability Case Study: Results

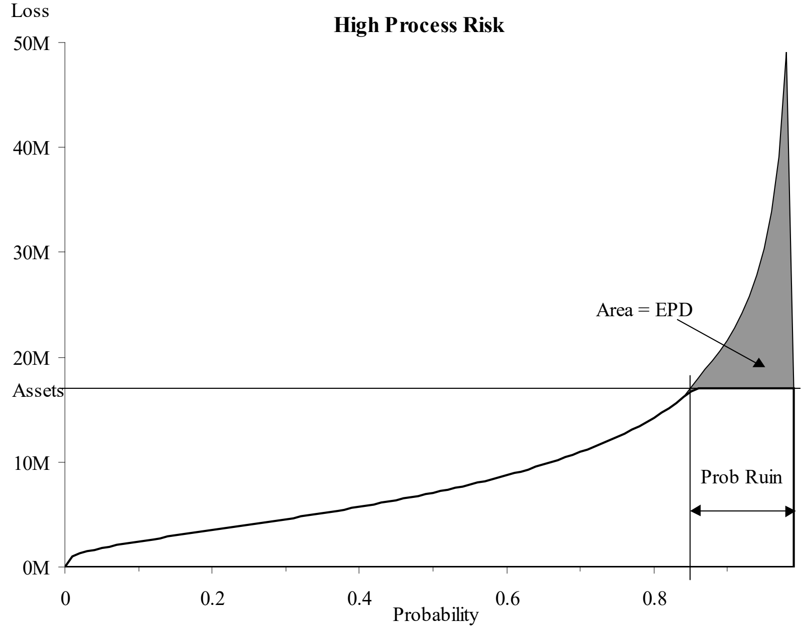
	No correlation			With Correlation		
EPD Level	Capital	P:S	L:S	Capital	P:S	L:S
1.0%	2.6M	5.0:1	6.9:1	3.7M	3.6:1	4.9:1
0.5%	3.9M	3.4:1	4.6:1	5.1M	2.5:1	3.4:1
0.1%	5.9M	2.2:1	3.1:1	8.2M	1.6:1	2.2:1

	No correlation			With Correlation		
Prob Ruin	Capital	P:S	L:S	Capital	P:S	L:S
10.0%	3.9M	3.4:1	4.6:1	4.7M	2.8:1	3.8:1
1.0%	7.8M	1.7:1	2.3:1	9.4M	1.4:1	1.9:1
0.1%	11.0M	1.2:1	1.6:1	13.0M	1.0:1	1.4:1

DFA Liability Case Study

- Comments
  - Probability of ruin, not EPD, drives capital requirements for low process risk lines





DFA Liability Case Study

- Comments
  - Using outstanding liabilities as denominator doubles indicated EPD ratios
  - Paper by Phillips estimates industry EPD at 0.15%
    - http://rmictr.gsu.edu/ctr/working.htm, #95.2
  - Correlation used following matrix

AY 1995	1.0	0.3	0.2	0.1
AY 1996	0.3	1.0	0.3	0.2
AY 1997	0.2	0.3	1.0	0.3
AY 1998	0.1	0.2	0.3	1.0

  - Model shows significant impact of correlation on required capital

Summary

- Use simulation carefully
  - Alternative methods of numerical integration
  - Concentrate simulated points in area of interest
- Iman-Conover provides powerful method for modeling correlation
- Use Fast Fourier Transforms to add independent random variables
- Consider annual statement data and use of statistical models to help calibrate DFA